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Massive vacua of $\mathcal{N} = 1^*$ theory and S-duality from matrix models

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Abstract: In this note we show how Dijkgraaf and Vafa's hypothesis relating the exact superpotential of an $\mathcal{N} = 1$ theory to a matrix model can be used to describe all the massive vacua of the $\mathcal{N} = 1^*$, or mass deformed $\mathcal{N} = 4$, theory including the Higgs vacuum. The matrix model computation of the superpotential for each massive vacuum independently yields a modular function of the associated effective coupling in that vacuum which agrees with previously derived results up to a vacuum-independent additive constant. The results in the different massive vacua can be related by the action of $\mathrm{SL}(2, \mathbb{Z})$ on the $\mathcal{N} = 4$ coupling, thus providing evidence for modular invariance of the underlying $\mathcal{N} = 4$ theory.

Keywords: Nonperturbative Effects, 1/N Expansion, Supersymmetry and Duality, Matrix Models.

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Contents

1. Introduction

According to Dijkgraaf and Vafa's (DV) proposal [1, 2, 3, 4], the effective superpotentials for a large class of $\mathcal{N}=1$ supersymmetric field theories at *finite* N are computed from related large-N matrix models. Some of the best tests of this hypothesis have been for certain mass deformations of $\mathcal{N}=4$ supersymmetric gauge theory which preserve $\mathcal{N}=1$ supersymmetry — the so-called $\mathcal{N}=1^*$ theory. The latter theory is obtained by adding masses for all three adjoint chiral superfields of the $\mathcal{N}=4$ theory. In addition, one can allow arbitrary polynomial deformations of the superpotential involving one of the three adjoint superfields. In this note, we will refer to this entire class of theories as $\mathcal{N}=1^*$ theories. These theories provide some of the best tests of the DV proposal, since the resulting effective superpotential depends in a non-trivial and characteristic way on the coupling constant of the $\mathcal{N}=4$ theory. Up till now the checks have involved calculating the effective superpotential in the confining vacuum for the quadratic $\mathrm{Tr}\,\Phi^2$ deformation [4] and then for arbitrary polynomial deformations and holomorphic condensates in the confining vacuum [5]. In the present note we extend this analysis to cover all the massive vacua of the theory, including the Higgs vacuum.

We find that the result of the $\mathrm{Tr}\,\Phi^2$ deformation is an effective superpotential in each massive vacuum of the $\mathcal{N}=1^*$ theory computed using the DV prescription from the matrix model of the form

$$W_{\mathrm{eff}}=-\frac{Np^2}{12}E_2\left(\frac{p(p\tau+k)}{N}\right),\qquad k=0,1,\ldots,\frac{N}{p}-1\,. \tag{1.1}$$

where p is a divisor of N. Once the vacuum-independent constant $N^2E_2(\tau)/12$ is added, the results in different vacua are related to each other by the action of the modular group, $\mathrm{SL}(2,\mathbb{Z})$ on τ, the $\mathcal{N}=4$ coupling, and the results agree perfectly with earlier computations using different approaches [6, 8]. What is remarkable about the derivation from the matrix model is that the modular properties emerge as a result of the computation without being assumed at the beginning.

Let us first briefly recall the vacuum structure of the $\mathcal{N}=1^*$ theory. In $\mathcal{N}=1$ language, the $\mathcal{N}=4$ $\mathrm{SU}(N)$ theory with coupling constant $\tau\equiv 4\pi i/g^2_{\mathrm{YM}}+\theta/2\pi$ has three

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adjoint chiral superfields Φ^+, Φ^- and Φ. We consider a general class of deformations of the $\mathcal{N}=4$ theory specified by a tree level superpotential

$$W = \frac{1}{g_{\rm YM}^2}\,\mathrm{Tr}\left(i\Phi[\Phi^+,\Phi^-] + \ \Phi^+\Phi^- + V(\Phi)\right), \tag{1.2}$$

where $V(\Phi)$ is a general polynomial

$$V(\Phi) \equiv \sum_p g_p\,\Phi^p\,. \tag{1.3}$$

Note that for the sake of simplicity we have set all masses to unity in the knowledge that it is simple to re-introduce them. For $V(\Phi)=\Phi^2$ we obtain the basic $\mathcal{N}=1^*$ theory, however, it is also interesting to consider the space of possible $\mathcal{N}=1^*$ deformations as in [5].

The vacuum structure can be deduced by proceeding with a classical argument augmented by some quantum considerations. The classical vacuum structure follows by solving the F- and D-flatness conditions modulo gauge transformations. Equivalently, one solves the F-flatness conditions modulo complex gauge transformations. Up to complex gauge transformations, the solutions of the F-flatness conditions are associated to representations of SU(2) in the following way: Φ is precisely iJ_3 of an SU(2) representation, reducible or irreducible, of dimension N, and $\Phi^\pm$ have the same non-zero elements as $J^\pm$ except that the actual numerical values depend on the form of the potential $V(\Phi)$. Vacua can be classified according to whether there are, after the Higgs mechanism, unbroken U(1)'s, in which case it is a massless vacuum. On the contrary, if the unbroken gauge group is non-abelian (or empty) one has a massive vacuum. In terms of SU(2) representations, the massive vacua correspond to N/p copies of the p-dimensional representation, where, by construction, p must be a divisor of N. The unbroken gauge group is then $\mathrm{SU}(N/p)$. The classical eigenvalues of Φ are explicitly

$$\lambda_j^{\rm cl} = \frac{i}{2}(p-2j+1)\,,\qquad j=1,\ldots,p\,, \tag{1.4}$$

each with a degeneracy of N/p. Now quantum reasoning is needed to deduce the multiplicity of vacua for each p. One expects at low energies that the theory flows to pure $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $\mathrm{SU}(N/p)$. Standard arguments then imply there are N/p physically inequivalent quantum vacua. All-in-all, there are $\sum_{p|N} N/p \equiv \sum_{p|N} p$ massive vacua, where p is a divisor of N.

It is the goal of this paper to calculate (1.1) from the matrix model proposal of Dijkgraaf and Vafa [1, 2, 3, 4], generalizing the results of [4, 5], which held for the confining vacuum ($p=1$ above).

2. The matrix model and its solution

As a direct consequence of the proposal of Dijkgraaf and Vafa [1, 2, 3, 4], in a given vacuum the exact effective superpotential for the above class of deformations of the $\mathcal{N}=4$ theory

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is computed by the planar diagram expansion, i.e. a large-N limit, of the matrix model for which there is a matrix variable for each chiral multiplet of the field theory and whose action is the tree-level superpotential of the field theory. Using the same notation for the matrix variables and their associated superfields, the matrix model partition function is

$$Z = \int [d\Phi^+][d\Phi^-][d\Phi] \exp -\frac{1}{g_s} \operatorname{Tr} \left(i\Phi[\Phi^+, \Phi^-] + \Phi^+\Phi^- + V(\Phi) \right) . \tag{2.1}$$

In the matrix model, unlike in the field theory, one takes $\Phi^+ = (\Phi^-)^\dagger$ and the fluctuations of Φ around the ensuing saddle points to be hermitean. At the saddle-point we will allow the eigenvalues of Φ to be complex. As noted for example in [10], the key fact that permits the solution of these matrix models is that one may integrate out $\Phi^\pm$ exactly to obtain a one-matrix integral to be solved in the large-N limit:

$$Z = \int [d\Phi] \frac{e^{-\frac{1}{g_s} \operatorname{Tr} V(\Phi)}}{\det(\mathrm{Adj}_\Phi + i)} . \tag{2.2}$$

The resulting one-matrix model actually becomes tractable in the large-N limit by going to the eigenvalue basis and performing a large-N saddle-point approximation to the integral. The details of this procedure have been extensively discussed in the literature. We refer the reader to [10], and references therein, for more details.

In order to describe the confining vacuum, for which $\Phi = 0$ classically, one takes a one cut solution as follows [4, 5]. The eigenvalues λ_j of Φ interact via a repulsive effective potential and form a continuum in the large-N limit and condense onto a cut along the real axis. The actual extent of the cut and the density of eigenvalues $\rho(\lambda)$ along the cut is self-consistently determined by the saddle-point equation in terms of the parameters of the deformation $V(\Phi)$ and the matrix model 't Hooft coupling $S = g_s N$. Proceeding along these lines leads to a description of the confining vacuum [4, 5].

For a general massive vacuum associated in the nomenclature of SU(2) representations to N/p copies of the p-dimensional representation, the classical eigenvalues of Φ are given in (1.4). This suggests in the matrix model we should take a multi-cut solution where at large N the eigenvalues condense on p cuts defined as

$$\mathcal{C}_j = \left\{ \lambda = \lambda_j^{\mathrm{cl}} + x \,, \ -\alpha_j \le x \le \alpha_j \right\} , \qquad j = 1, \ldots, p . \tag{2.3}$$

The saddle-point equation in the large-N limit is most conveniently written in terms of the resolvent function

$$\omega(z) = \int \frac{\rho(\lambda)}{z - \lambda} d\lambda \tag{2.4}$$

where $\rho(\lambda)$ is the unit normalized spectral density of eigenvalues:

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) , \tag{2.5}$$

a function that in the large-N limit only has non-trivial support on the cuts. The spectral density is normalized in such a way that the filling fractions of eigenvalues along each cut

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are given by

$$\frac{N_j}{N} = \int_{-\alpha_j}^{\alpha_j} \rho\big(\lambda_j^{\rm cl} + x\big)\, dx\,. \tag{2.6}$$

where N_j denotes the number of eigenvalues along the jth cut.

The resolvent $\omega(z)$ is an analytic function on the complex z-plane which has cuts precisely along each $\mathcal{C}_j$, $j = 1, \ldots, p$. The discontinuity across each cut gives the spectral density at that point:

$$\omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon) = -2\pi i \rho(\lambda); \qquad \lambda \in \mathcal{C}_j\,. \tag{2.7}$$

In terms of $\omega(z)$, the saddle-point equation is

$$\begin{aligned} \frac{V'(\lambda)}{S} &= \omega(\lambda + i\epsilon) + \omega(\lambda - i\epsilon) - \\ &\quad -\frac{1}{2}\big(\omega(\lambda + i + i\epsilon) + \omega(\lambda + i - i\epsilon) + \omega(\lambda - i + i\epsilon) + \omega(\lambda - i - i\epsilon)\big) \end{aligned} \tag{2.8}$$

for $\lambda \in \cup_j \mathcal{C}_j$. Notice that various principal values have been taken, giving rise to the $\pm i\epsilon$ prescriptions, in order to render the integrals well defined. Unlike the single cut solution, the final terms — those shifted by $\pm i$ — also have to be defined as a principal value since most of the shifted cuts collide with each other in pairs.

At this point we need to employ some extra guesswork. We start with the observation that in the massive vacuum there should be same number of eigenvalues on each cut. In other words the filling fractions (2.6) should be the same. This is well motivated by the classical solutions for massive vacua of the *field theory* wherein the adjoint scalars have VEVs that split into N/p copies of a p-dimensional representation of SU(2) preserving an SU(N/p) gauge symmetry classically. Of course, we would also expect solutions of the saddle-point equations with different filling fractions which would correspond to massless vacua. However, we are going to specialize to the massive vacua by making an ansatz which automatically ensures that the filling fractions are identical. The ansatz we make is that the eigenvalue density is the same along each of the cuts (all of which have the same extent $\alpha_j = \alpha$): for each pair j, k

$$\rho\big(\lambda_j^{\rm cl} + x\big) = \rho\big(\lambda_k^{\rm cl} + x\big)\,, \qquad -\alpha \le x \le \alpha\,, \tag{2.9}$$

or equivalently

$$\omega(\lambda_j^{\rm cl} + x + i\epsilon) - \omega(\lambda_j^{\rm cl} + x - i\epsilon) = \omega(\lambda_k^{\rm cl} + x + i\epsilon) - \omega(\lambda_k^{\rm cl} + x - i\epsilon)\,. \tag{2.10}$$

Having made this ansatz we now investigate the saddle-point equations (2.8). Notice that they are linear in the $w(\lambda + i\epsilon) + w(\lambda - i\epsilon)$ and so we can solve for these quantities at each cut. One finds

$$\begin{aligned} \omega(\lambda_j^{\rm cl} + x + i\epsilon) + \omega(\lambda_j^{\rm cl} + x - i\epsilon) \,&=\, 2Y(\lambda_j^{\rm cl} + x) + \frac{2}{p+1}\bigg((p+1-j)\omega\Big(x + \frac{i(p+1)}{2}\Big) + \\ &\qquad + j\omega\Big(x - \frac{i(p+1)}{2}\Big)\bigg)\,, \end{aligned} \tag{2.11}$$

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where $-\alpha \le x \le \alpha$ and $j = 1, \ldots, p$. Notice that the resolvent has no discontinuities at $x \pm i(p+1)/2$. In the above, $Y(\lambda)$ is the polynomial which is determined by the potential $V(\lambda)$ via

$$\frac{V'(\lambda)}{S} = 2Y(\lambda) - Y(\lambda + i) - Y(\lambda - i)\,. \tag{2.12}$$

As in the one-cut solution, it useful to define the function

$$G(z) = U(z) + iS\left(\omega\left(z + \frac{i}{2}\right) - \omega\left(z - \frac{i}{2}\right)\right) \tag{2.13}$$

where $U(z)$ is a polynomial in z such that

$$iV'(z) = U\left(z + \frac{i}{2}\right) - U\left(z - \frac{i}{2}\right). \tag{2.14}$$

Given the ansatz (2.9), the analytic structure of $G(z)$ is much simpler than that of the resolvent itself. The cuts of $\omega(z)$ are shifted up and down by $\pm i/2$ and because of (2.10) most cancel pairwise. There are only two remaining cuts — the ones on the outside of the stack — given by $\frac{i}{2}p + x$ and $-\frac{i}{2}p + x$, respectively, for $-\alpha \le x \le \alpha$. Furthermore, putting (2.11) and (2.10) together, we end up with a rather simple and suggestive constraint on $G(z)$:

$$G\left(x + \frac{i}{2}p \pm i\epsilon\right) = G\left(x - \frac{i}{2}p \mp i\epsilon\right); \qquad -\alpha \le x \le \alpha\,. \tag{2.15}$$

Now we see that having made the ansatz we now have what is closely related to the one-cut solution, $p = 1$, of [4, 5]. In terms of the function $G(z)$, the only difference is that the two cuts of $G(z)$ are shifted from $\pm i/2$ to $\pm ip/2$. Furthermore, just as in the $p = 1$ case, the equation (2.15) has the effect of gluing the top (bottom) of the upper cut to the bottom (top) of the lower cut making what is effectively a torus. This is very important, since on general grounds we expect each of the massive vacua to be associated to a particular torus with its own complex structure [6, 7, 11, 12].

Fortunately, we can now almost lift the analysis of the one-cut solution, with only slight modifications to account for the new positions of the cuts, and we refer to [5] for more details and conventions. Firstly, we can uniformize the torus in the u-plane by means of the mapping

$$z(u) = -\frac{p\omega_1}{\pi}\left[\zeta(u) - \frac{\zeta(\omega_1)}{\omega_1}u\right] \equiv -\frac{p}{2}\frac{\theta_1'(\frac{\pi u}{2\omega_1}|\tilde\tau)}{\theta_1(\frac{\pi u}{2\omega_1}|\tilde\tau)}\,. \tag{2.16}$$

The torus is defined by the quotient of the complex u-plane by the lattice generated by the two complex numbers $2\omega_1$ and $2\omega_2$. The complex structure of the torus $\tilde\tau = \omega_2/\omega_1$ is a parameter. In effect we shall see that we have changed variables from α — and hence from S — to the new variable $\tilde\tau$. Notice in (2.16), the multiplicative factor of p which ensures that the cuts occur in the requisite positions. Tightly encircling around the upper (lower) cut anti-clockwise in the z-plane corresponds to $\omega_2 + 2x\omega_1$ (or $-\omega_2 + 2(1-x)\omega_1$) for $0 \le x \le 1$ on the u-plane. The former, or minus the latter, defines the A-cycle of the torus. In addition, the point at infinity in the z plane corresponds to $u = 0$.

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At this point we find it convenient to choose the simplest $\mathcal{N}=1^*$ deformation obtained by taking a potential $V(\Phi)=\Phi^2$, although it should be said that more complicated potentials are just as tractable [5]. In this case, eq. (2.13) fixes the asymptotic form of G to be

$$G(z) \overset{z\to\infty}{=} z^2+\mathcal{O}\left(\frac{1}{z^2}\right) . \tag{2.17}$$

Given the fact that G must be single valued on the torus — an elliptic function — this fixes its form uniquely. Eq. (2.17) implies that in the u plane $G(z(u))$ must have a double pole at $u=0$ with a set coefficient. Given that (2.16) implies

$$u=-\frac{p\omega_1}{\pi}\frac{1}{z}+\frac{p^3\omega_1^2\zeta(\omega_1)}{\pi^3}\frac{1}{z^3}+\cdots , \tag{2.18}$$

means that

$$G(z(u))=\frac{p^2\omega_1^2}{\pi^2}\left[\wp(u)-\frac{2\zeta(\omega_1)}{\omega_1}\right] . \tag{2.19}$$

Once again take note of the factor of p^2 which will prove important.

We are now ready to implement the Dijkgraaf-Vafa prescription. First of all, recall that the filling fractions for each of the p original cuts are given as in eq. (2.6). For our ansatz (2.9) each of the filling fractions are equal and following the analysis in [4, 5] we can express them as an integral of $G(z(u))$ around the A-cycle of the torus:

$$2\pi i S_j=-i\int_A G(z(u))\frac{dz(u)}{du}du=\frac{p^3}{12}\frac{dE_2(\tilde{\tau})}{d\tilde{\tau}} , \qquad S_j\equiv g_s N_j . \tag{2.20}$$

The factor of p^3 arising from the p^2 and p factors of $G(z)$ and $z(u)$, respectively, will turn out to be crucial. In the above E_2 is the 2^{nd} Eisenstein series (see [5] for definitions).

The other important ingredient in the Dijkgraaf-Vafa proposal is the quantity $\partial\mathcal{F}_0/\partial S_j$ being the variation in the genus zero free energy of the matrix model upon transporting an eigenvalue in from infinity to the j^{th} cut to infinity. In the present context where we will take all the filling fractions to be the same, we only need an expression for the sum $\sum_{j=1}^p \partial\mathcal{F}_0/\partial S_j$. This quantity is then simply related to an integral of the same form as (2.20), but now over the B cycle of the torus $-\omega_1+(2x-1)\omega_2$, $0\leq x\leq 1$. The integral is easily evaluated following [5]

$$\sum_{j=1}^p\frac{\partial\mathcal{F}_0}{\partial S_j}=-i\int_B G(z(u))\frac{dz(u)}{du}du=p^3\left(\frac{\tilde{\tau}}{12}\frac{dE_2(\tilde{\tau})}{d\tilde{\tau}}-\frac{1}{12}E_2(\tilde{\tau})\right) . \tag{2.21}$$

Again the factors of p are crucial: $G(z)$ and $z(u)$ give p^2 and p respectively.

The effective superpotential is obtained by extremizing the following expression with respect to S_j:

$$W_{\text{eff}}=\sum_{j=1}^p\left(N_j\frac{\partial\mathcal{F}_0}{\partial S_j}-2\pi i\tau S_j\right) . \tag{2.22}$$

In this expression N_j are the number of eigenvalues of Φ which are classically equal to λ_j^{cl}. It is important to realize that these are not directly related to the N_j in the matrix

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model (although we use the same notation). The latter are related to the filling fractions S_j/S via $N_j = S_j/g_s$. In order to describe the massive vacua we now take the physical $N_j = N/p$. Since N_j is an integer, p must be a divisor of N. In this case, after having made the ansatz (2.9), which solves the saddle-point equations, each of S_j depends upon only one parameter, namely $\tilde{\tau}$, the complex structure of the auxiliary torus in the u-plane. Hence, in implementing the Dijkgraaf-Vafa prescription for the massive vacua, we only have to extremize the superpotential with respect to $\tilde{\tau}$. Substituting our expressions (2.20) and (2.21), we have

$$W_{\rm eff} = Np^2 \left(\frac{\partial \mathcal{F}_0}{\partial S}\right)_{p=1} - 2\pi i \tau p^4 S_{p=1}\,. \tag{2.23}$$

where we have written the result in terms terms of the ($p=1$) one-cut solution quantities in order to highlight the p dependence:

$$2\pi i S_{p=1} = \frac{1}{12}\frac{dE_2(\tilde{\tau})}{d\tilde{\tau}}\,, \qquad \left(\frac{\partial \mathcal{F}_0}{\partial S}\right)_{p=1} = \frac{\tilde{\tau}}{12}\frac{dE_2(\tilde{\tau})}{d\tilde{\tau}} - \frac{1}{12}E_2(\tilde{\tau})\,. \tag{2.24}$$

It is easy to see that

$$\frac{\partial W_{\rm eff}}{\partial \tilde{\tau}} = 0 \Longrightarrow \tilde{\tau} = \frac{p^2\tau}{N}\,. \tag{2.25}$$

and substituting back we have

$$W_{\rm eff} = -\frac{Np^2}{12}E_2\left(\frac{p^2\tau}{N}\right). \tag{2.26}$$

where as stated above p has to be a divisor of N. Upon restoring the mass scales in the problem this result agrees precisely, up to a vacuum-independent additive constant, with the expressions of [6, 7]. Note that $\mathrm{SL}(2,\mathbb{Z})$ modular transformations of τ relate the superpotentials in different massive vacua with the understanding that $W_{\rm eff}$ has modular weight 2 and that one must account for the anomalous modular transformation of E_2 by adding the vacuum independent constant $N^2E_2(\tau)/12$.[1] This is remarkable in that S-duality of the underlying $\mathcal{N}=4$ theory was not assumed to begin with, but rather emerges from the solution of the matrix model.

We want to emphasize that our method also covers the Higgs vacuum (and other vacua where p scales with N) which at first sight is puzzling because in the field theory all the eigenvalues are — classically at least — non-degenerate. How can they form continua on N separate cuts in the matrix model when there are only N eigenvalues? The resolution of this puzzle is the realization that the N_j in the field theory are not equated to the $N_j = S_j/g_s$ in the matrix model. The latter can be large while the former can be set to one in (2.22).

Notice that our result (2.26) only describes one of the N/p massive vacua associated in the original analysis to the p-dimensional representation of SU(2). The other $N/p-1$ such vacua are obtained by repeatedly performing the modular transformation $\tau \to \tau+1$ a sufficient number of times to give (1.1).

[1] The freedom to add such a constant is an example of the more general effect of operator mixing (in this case Φ^2 with the identity) [8].

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Our multi-cut technology can also be employed in the $\mathcal{N}=1^*$ deformation of the Leigh-Strassler theory for which the confining vacuum result was quoted in [5]. The result in the $p^{\rm th}$ massive vacuum is

$$W_{\rm eff} = -\frac{N}{p}\frac{\cos(\beta/2)}{4\sin^3(\beta/2)}\frac{\theta_1(\beta p/2|p^2\tau/N)}{\theta_1'(\beta p/2|p^2\tau/N)} + \frac{N}{4\sin^2(\beta/2)}\,. \tag{2.27}$$

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